

String Motion in Fivebrane Geometry

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The classical motion of a test string in the transverse space of two types of heterotic fivebrane sources is fully analyzed, for arbitrary instanton scale size. The singular case is treated as a special case and does not arise in the continuous limit of zero instanton size. We find that the orbits are either circular or open, which is a solitonic analogy with the motion of an electron around a magnetic monopole, although the system we consider is quantitatively different. We emphasize that at long distance this geometry does not satisfy the inverse square law, but satisfies the inverse cubic law. If the fivebrane exists in nature and this structure survives after any proper compactification, this last result can be used to test classical “stringy” effects.

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1. Introduction

The structures of classical solitonic solutions of string theory have been actively investigated recently[1]. Among these solutions the heterotic fivebrane solution conjectured by Duff[2] and constructed by Strominger[3] is particularly interesting because it is dual to the fundamental string in the generalized sense of the electric-magnetic duality¹. Although such a duality does not necessarily imply the existence of the dual object, e.g. we have not yet found the magnetic monopole, it is worth while to further investigate the implications of this duality.

In this paper we follow the analogy of the electron-monopole system and shall study some of the classical motions of a test string around a fivebrane source. We shall restrict ourselves in this paper to the case in which both string and fivebrane behave like points in the transverse space of the fivebrane. For simplicity we also ignore any contribution due to world-sheet fermions. Nevertheless, even if we include this contribution, we expect that there will be no qualitative change in the dynamics because these fermions only couple to the instanton YM background.

We shall consider here both cases of “gauge” and “symmetric” solutions[5]. The symmetric solution is closely related to the elementary fivebrane solution of ref.[6]. Though the gauge solution is not an exact classical background, we still observe various interesting structures, which we expect will hold in a qualitatively similar way even for the exact fivebrane background. Since the symmetric solution is exact, we can perform an exact analysis.

In both cases, analogies are made between the string-fivebrane system and electron orbits around a monopole. For example, in both systems there exist circular orbits whose discretized radii are governed by the quantization of the source charge (the instanton charge in the case of the fivebrane and the magnetic charge in the case of the monopole). Also, both the fivebrane and the monopole impart an intrinsic angular momentum to the string and the electron respectively.

Since the dynamics of the string-fivebrane system is governed by the competition between the attractive gravitational force and the repulsive force due to the antisymmetric tensor field, we are led to expect fundamental differences with General Relativity, in which

¹ This duality which interchanges Noether charge (e.g. electric charge) and topological charge (e.g. monopole charge) is in principle the foundation for the Montonen-Olive conjecture[4], which is yet to be confirmed rigorously.

the latter force is absent. This expectation is indeed confirmed by our finding that the long-distance forces obey an inverse cubic law, rather than an inverse square law, in direct contrast to General Relativity. This finding could have important implications for physics according to string theory at scales larger than the Planck length, provided the fivebrane structure survives compactification.

This paper is organized as follows. In sect.2 we shall review the derivation of both solutions. In sect.3 we describe the dynamics generically, for an arbitrary instanton scale size. In sect.4, we choose a finite instanton scale size and to leading order in instanton charge, we study perturbatively the case of the gauge solution. In sect.5, setting the instanton size to zero, we describe the symmetric solution case. In both cases we observe drastic differences from General Relativity or Newtonian dynamics. Finally, in sect.6 we shall provide further perspectives.

2. Heterotic Fivebranes

Let us review the derivations given in refs.[3][5]. The heterotic fivebrane is a solution to the equations of the supersymmetric vacuum for the heterotic string

$$\delta\psi_M = \left(\nabla_M - \frac{1}{4}H_{MAB}\Gamma^{AB}\right)\epsilon = 0, \quad (2.1)$$

$$\delta\lambda = \left(\Gamma^A\partial_A\phi - \frac{1}{6}H_{AMC}\Gamma^{ABC}\right)\epsilon = 0, \quad (2.2)$$

$$\delta\chi = F_{AB}\Gamma^{AB}\epsilon = 0, \quad (2.3)$$

where ψ_M , λ and χ are the gravitino, dilatino and gaugino, while

$$dH = \alpha' \left(\text{tr}R \wedge R - \frac{1}{30}\text{Tr}F \wedge F\right). \quad (2.4)$$

In the above we have properly rescaled all the field variables so that the string coupling $g_s = e^\phi$ and α' are the only independent couplings. In the heterotic string α' is proportional to κ^2/g_{YM} , where κ is the gravitational coupling constant and g_{YM} is the YM coupling constant. Depending on the structure of eq.(2.4), we can have different types of solutions. First, we shall review the derivation of the so-called “gauge” solution and then later the “symmetric” solution.

In (1+9)-dimension we have Majorana-Weyl fermions, which decompose down to chiral spinors according to $\text{SO}(1,9) \supset \text{SO}(1,5) \otimes \text{SO}(4)$ for the $M^{1,9} \rightarrow M^{1,5} \times M^4$ decomposition. For such spinors the dilatino equation (2.2) is satisfied by

$$H_{\mu\nu\lambda} = \pm \epsilon_{\mu\nu\lambda\sigma} \partial^\sigma \phi, \quad (2.5)$$

where μ, ν, \dots are indices for the transverse space M^4 and $\phi = \phi(x^\mu)$, while we shall use indices a, b, \dots for $M^{1,5}$ below.

The other equations are solved by constant chiral spinors ϵ_\pm and

$$g_{ab} = \eta_{ab}, \quad g_{\mu\nu} = e^{2\phi} \delta_{\mu\nu} \quad (2.6)$$

such that

$$\begin{aligned} \delta\psi_\mu &= \left(\nabla_\mu - \frac{1}{2} \Gamma^{\mu\nu} \partial_\nu \phi \right) \epsilon_\pm = \partial_\mu \epsilon_\pm = 0, \\ \delta\psi_a &= \nabla_a \epsilon_\pm = \partial_a \epsilon_\pm = 0, \end{aligned} \quad (2.7)$$

and

$$\delta\chi = F_{\mu\nu}^\pm \Gamma^{\mu\nu} \epsilon_\pm = -F_{\mu\nu}^\pm \Gamma^{\mu\nu} \epsilon_\pm = 0, \quad (2.8)$$

where eq.(2.8) is satisfied using an instanton configuration for the (anti)self-dual YM equation in M^4

$$F_{\mu\nu}^\pm = \pm \frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma}^\pm \quad (2.9)$$

for an $\text{SU}(2)$ subgroup of $E_8 \times E_8$ or $\text{SO}(32)$. In fact $\phi = \phi(r^2)$ here (i.e. no angular dependence), where $r^2 = \sum (x^\mu)^2$. With a finite instanton scale size λ ,

$$e^{2\phi} = e^{2\phi_0} + Q \frac{(r^2 + 2\lambda^2)}{(r^2 + \lambda^2)^2}, \quad (2.10)$$

where ϕ_0 is the value of the dilaton at spatial infinity and Q is the charge of the instanton in the unit of α' , e.g. $Q = 8\alpha'$.

Note that this solution shows a scale symmetry

$$\begin{aligned} \phi &\rightarrow \phi + \ln \sigma, \\ r &\rightarrow \sigma^{-1} r, \\ \lambda &\rightarrow \sigma^{-1} \lambda, \end{aligned} \quad (2.11)$$

where σ is a constant. The $\lambda = 0$ case is not related to the $\lambda \neq 0$ case in terms of this scale symmetry, but it retains a similar scale symmetry without the last property.

Due to such a scale symmetry, when we later describe the dynamics of the system, particularly in terms of the effective potential, we need to modify its form slightly in order to secure the symmetry. Furthermore, there are certain limits of these solutions which are not related by the above scale symmetry. We thus prefer keeping ϕ_0 explicitly instead of fixing its value using the scale symmetry. For example, $\phi_0 = -\infty$ is one extreme limit not related by this scale symmetry.

In the low energy effective action for the heterotic string the potential for the dilaton ϕ is identically zero if the scale symmetry of the theory is respected. This scale symmetry is expected to be spontaneously broken to determine some physically relevant dilaton vacuum expectation value. For any nontrivial potential for ϕ there is always one rather trivial minimum at $\phi \rightarrow -\infty$, which can be identified as a boundary value of ϕ , so that $\phi_0 \rightarrow -\infty$. Note that at this point it is clear that the previously mentioned scale symmetry is spontaneously broken. Though this minimum point is not really relevant to the low energy physics unless we find another isolated minimum, it is still worth while to single out this case.

We now have a fivebrane living in $M^{1,5}$ which is a point-like object in M^4 . The above solution is called the “gauge” solution[5], while there is another “symmetric” fivebrane solution which is exact.

This symmetric solution can be obtained as follows. Define a generalized connection by

$$\Omega_{\pm M}^{AB} = \omega_M^{AB} \pm H_M^{AB} \quad (2.12)$$

embedded in the $SU(2)$ subgroup and equate it to the gauge connection A_μ so that $dH = 0$ and the corresponding curvature $R(\Omega_\pm)$ cancels against the Yang-Mills field strength F [7]. It follows that

$$R(\Omega_\pm)_{\mu\nu}^{mn} = \mp \frac{1}{2} \epsilon_{\mu\nu}^{\lambda\sigma} R(\Omega_\pm)_{\lambda\sigma}^{mn}. \quad (2.13)$$

Thus we have a solution

$$\begin{aligned} e^{2\phi} &= e^{2\phi_0} + \frac{Q}{r^2}, \\ H_{\mu\nu\lambda} &= -\epsilon_{\mu\nu\lambda\sigma} \partial^\sigma \phi, \\ F_{\mu\nu}^{mn} &= R(\Omega_-)_{\mu\nu}^{mn}, \end{aligned} \quad (2.14)$$

where both F and R are (anti)self-dual.

This symmetric solution becomes exact since $A_\mu = \Omega_{-\mu}$ implies that all the higher order corrections vanish[8].

3. Generic Case

For the general case we keep the instanton size λ arbitrary and shall treat generically both “gauge” and “symmetric” solutions. For the former we need only know the solution to leading order in α' , but for the latter we use the exact expression with $\lambda = 0$. Naively one may associate the symmetric solution with the $\lambda \rightarrow 0$ limit of the gauge solution in such a way that the higher order corrections vanish as $\lambda \rightarrow 0$. However, we shall later find out that the formal similarity of these solutions does not necessarily imply the same limit in the dynamics². This could be anticipated in some sense by observing that the symmetric solution only makes sense for $\lambda \gg \sqrt{\alpha'}[5]$ so that the limit $\lambda \rightarrow 0$ does not in fact make sense.

The Lagrangian for a string moving in a given background of massless fields is given by

$$\mathcal{L} = \frac{1}{2} \left(\sqrt{-\gamma} \gamma^{ij} \partial_i X^M \partial_j X^N g_{MN} + \epsilon^{ij} \partial_i X^M \partial_j X^N B_{MN} \right) + \dots, \quad (3.1)$$

where γ_{ij} is the worldsheet metric for $(i, j) = (\tau, x)$, g_{MN} is the “ σ -model metric” and \dots includes the worldsheet fermion terms, which we ignore for simplicity. Throughout this paper, we assume that the string is parallel to one of the fivebrane directions, i.e. x will be identified as one of the fivebrane coordinates. Since B_{MN} is only nonzero when both M and N are transverse, the axion does not contribute to the Lagrangian of the point-like string in the transverse space. One can put the test string inside the transverse space, but for our purposes this is not really necessary, unless we identify the $(1+3)$ -dimensional subspace in order to study the dynamics inside “space-time”. We will leave this for a future study.

If we substitute the worldsheet constraint equation $\gamma_{ij} = \partial_i X^M \partial_j X^N g_{MN}$ in (3.1), the relevant Lagrangian for the classical dynamics of the string in the fivebrane background is simply of the Nambu-Goto type. From (2.6) it follows that

$$\mathcal{L} = \sqrt{-\gamma} = \left[\dot{t}^2 - e^{2\phi} \left\{ \dot{r}^2 + r^2 \left(\dot{\chi}^2 + \sin^2 \chi (\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2) \right) \right\} \right]^{1/2}, \quad (3.2)$$

where “ \cdot ” is the derivative with respect to the proper time τ . In contrast to the usual motion of a string in a given background (e.g. the motion of a cosmic string), we do not see any derivative along the string direction (i.e. x -direction) because we have identified x with

² We would like to call the reader’s attention to the fact that in the symmetric solution case the actual instanton size is not $\lambda = 0$ but Q . Thus the two solutions are not gauge equivalent.

the coordinate outside the transverse space to make the string point-like here. Note that the string dynamics is generically different from General Relativity, in which the geodesic equations are described by, say, the square of the above Lagrangian.

Though there is no explicit coupling between the string and the antisymmetric tensor field, the fivebrane geometry originates more or less from the rank three tensor $H_{\mu\nu\lambda}$. We should therefore expect that the dynamics will be governed by the competition between the attractive force due to gravity and the repulsive force due to the antisymmetric tensor. In fact we will see later that while the repulsive force dominates at long distance, there exists an attractive force region near the center of the fivebrane. It is noteworthy that this result deviates drastically from General Relativity.

Using the four-dimensional spherical symmetry of the Lagrangian, we can fix χ and θ . For simplicity, we take $\chi = \theta = \pi/2$ so that the problem reduces effectively to a two-dimensional one described by polar coordinates (r, φ) with a simplified Lagrangian

$$\mathcal{L} = (\dot{t}^2 - e^{2\phi}(\dot{r}^2 + r^2\dot{\varphi}^2))^{1/2}. \quad (3.3)$$

For time-like geodesics, $\mathcal{L} = 1^3$.

This system has two constants of motion along the time-like geodesics. One is “energy”

$$E \equiv \frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{\dot{t}}{\mathcal{L}} = \dot{t}, \quad (3.4)$$

where $E \geq 1$ can be interpreted as the conserved energy per unit mass of the string and represents a constant redshift. The other is “angular momentum”

$$L \equiv -\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{e^{2\phi} r^2 \dot{\varphi}}{\mathcal{L}} = e^{2\phi} r^2 \dot{\varphi}. \quad (3.5)$$

L can be interpreted as the conserved angular momentum of the string per unit mass and may be rewritten in the form $L = L_0 + Ql$. Recall that the dependence of $e^{2\phi}$ on the instanton solution eq.(2.10) tells us that the angular momentum is related to the instanton

³ Note that there is no null geodesic for the Nambu-Goto action induced from the nonzero string tension Polyakov action because $\gamma = 0$ implies that the induced worldsheet metric γ_{ij} is singular so that it cannot be inverted to define the original worldsheet Polyakov string action. The story may be different for the zero string tension case. From the pure Nambu-Goto action’s point of view, one can certainly allow null geodesics, but it now takes an infinite amount of “energy” to move along such a null geodesic so that each point behaves like a “black hole” with zero-radius horizon.

charge. This is analogous to the situation of the electron-monopole system, in which the angular momentum of the electron shifts due to the monopole charge. Thus the first term L_0 represents the free angular momentum, while the second term Ql , which is proportional to Q , represents the intrinsic angular momentum provided by the fivebrane. In this case, however, the motion of the string is restricted to its initial plane. Indeed we shall find that some of the characteristic motions of the test string around the fivebrane are analogous to the motions of an electron around a monopole source[9].

One more comment on the energy: E does not in fact scale as an energy with respect to the length scale for r but $\tilde{E} \equiv Ee^{-\phi_0}$ does. In order to define the “effective potential” V we rewrite the geodesic condition $\mathcal{L} = 1$ in the form

$$\dot{r}^2 + V^2 = \tilde{E}^2. \quad (3.6)$$

For $\phi_0 \rightarrow -\infty$, the scale symmetry is missing as we explained before and we can define the effective potential from $\dot{r}^2 + V^2 = E^2$ now, where V^2 can be easily determined. However, the behavior of V^2 seems to be rather unrealistic in most cases except the symmetric solution case, where for $a^2 > 0$ V^2 has a minimum at $r^2 = 0$ and is then monotonically increasing. This can be interpreted as a “confinement” of the test string at the fivebrane source point.

From eqs.(3.4)(3.5) and the geodesic condition we get

$$\dot{\phi} = \frac{L}{r^2 \left[e^{2\phi_0} + Q \frac{(r^2 + 2\lambda^2)}{(r^2 + \lambda^2)^2} \right]} \quad (3.7)$$

and

$$\dot{r}^2 = \frac{E^2 - 1}{e^{2\phi_0} + Q \frac{(r^2 + 2\lambda^2)}{(r^2 + \lambda^2)^2}} - \frac{L^2}{r^2 \left[e^{2\phi_0} + Q \frac{(r^2 + 2\lambda^2)}{(r^2 + \lambda^2)^2} \right]^2}. \quad (3.8)$$

From eq.(3.8) we can read off the effective potential

$$V^2 = E^2 e^{-2\phi_0} - \frac{E^2 - 1}{e^{2\phi_0} + Q \frac{(r^2 + 2\lambda^2)}{(r^2 + \lambda^2)^2}} + \frac{L^2}{r^2 \left[e^{2\phi_0} + Q \frac{(r^2 + 2\lambda^2)}{(r^2 + \lambda^2)^2} \right]^2}. \quad (3.9)$$

The turning points, if any, are found by setting $V = \tilde{E}$ (i.e. $\dot{r} = 0$).

If $E^2 = 1$, we have $\dot{r} = 0 = \dot{\phi}$ and $L = 0$ and the string remains stationary with no force acting on it. This is analogous to the electron-monopole case in the sense that if the electron is stationary, there is no force acting on it due to the monopole.

To determine the turning points for $E^2 > 1$ we must solve the following cubic equation for $x \equiv r^2$:

$$x^3 + (2\lambda^2 - \tilde{a}^2)x^2 + \lambda^2(\lambda^2 - 2\tilde{a}^2)x - \lambda^4(\tilde{a}^2 + \tilde{Q}) = 0, \quad (3.10)$$

where $\tilde{a} = ae^{-\phi_0}$, $a^2 \equiv \frac{L^2}{E^2-1} - Q$ and $\tilde{Q} = Qe^{-2\phi_0}$. One can easily see that this equation has at least one root at $\sqrt{x} = r_{\min} \geq 0$. In fact there is exactly one root in the physical region $x \geq 0$. The physical region is therefore restricted to $\sqrt{x} \geq r_{\min}$. In contrast to General Relativity, there are no bound orbits, except for a trivial unstable circular orbit at $r = r_{\min}$. This can be seen as the result of the dominance of the long range repulsive force over the long range attractive force of standard General Relativity.

The motion from the observer's point of view can be described by $d\varphi/dt = \dot{\varphi}/\dot{t} = \dot{\varphi}/E$ and $dr/dt = \dot{r}/E$. The asymptotic behavior of $\dot{\varphi}$ and \dot{r} is as follows: $\dot{\varphi} \rightarrow 0$, $\dot{r} \rightarrow \text{const.}$ as $r \rightarrow \infty$ and $\dot{\varphi} \rightarrow \text{const.}$, $\dot{r} \rightarrow 0$ as $r \rightarrow r_{\min}$.

The orbits can be classified according to the signs of \dot{r} , $\dot{\varphi}$ or dr/dt , $d\varphi/dt$; but it is always useful to take an analogy with the Keplerian analysis of Newtonian dynamics to describe the orbits. In this way we can also obtain a reference-frame independent classification of the orbits, although the situation here is quite different from that of the Keplerian orbits. As usual we introduce the variable $u \equiv 1/r$ for convenience.

For this purpose we first write down the orbit equation

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{E^2 - 1}{L^2} r^4 \left[e^{2\phi_0} + Q \frac{(r^2 + 2\lambda^2)}{(r^2 + \lambda^2)^2} \right] - r^2. \quad (3.11)$$

Note that, setting $dr/d\varphi = 0$, we recover the cubic equation eq.(3.10) for $L \neq 0$. In terms of u we obtain

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{E^2 - 1}{L^2} \left[e^{2\phi_0} + Q \frac{u^2 + 2\lambda^2 u^4}{(1 + \lambda^2 u^2)^2} \right] - u^2. \quad (3.12)$$

Later we shall find out that the solutions of this differential equation are in general not conic.

To see the competition between the attractive and repulsive forces we must compute the acceleration of the test string. The radial component is given by

$$a_r = Q \frac{r^2 + 3\lambda^2}{r(r^2 + \lambda^2)^3 \left[e^{2\phi_0} + Q \frac{(r^2 + 2\lambda^2)}{(r^2 + \lambda^2)^2} \right]^3} \left[(E^2 - 1)r^2 \left\{ e^{2\phi_0} + Q \frac{(r^2 + 2\lambda^2)}{(r^2 + \lambda^2)^2} \right\} - 2L^2 \right] \quad (3.13)$$

and the angular component is

$$a_\varphi = 2QL\dot{r} \frac{r^2 + 3\lambda^2}{\left[e^{2\phi_0} + Q \frac{(r^2 + 2\lambda^2)}{(r^2 + \lambda^2)^2} \right]^2 (r^2 + \lambda^2)^3}, \quad (3.14)$$

where \dot{r} is given by eq.(3.8). If we divide by E^2 , we can obtain the components of the acceleration measured by a far away observer. Needless to say, if Q vanishes, the acceleration vanishes. It is especially noteworthy that $a_r \propto Qr^{-3}$ as $r \rightarrow \infty$, as opposed to the usual long range inverse square law! We instead have an inverse cubic law⁴. If this structure survives after compactification to (1+3)-dimensional space-time, it could be used to test classical “stringy” effects on the dynamics.

Based on the sign of the radial component a_r , we can tell whether the force is attractive or repulsive. But, the characteristic features of the force depend on the angular momentum L since the attractive force can be generated only if there is angular motion. Generically, there are two regions. In one region (short distance) the attractive force dominates and in another region (long distance) the repulsive force dominates. The attractive force region shrinks as the scale size of the instanton becomes smaller. If $L = 0$, the attractive force region disappears. For $L \neq 0$, there are always two regions.

4. Gauge Solution

Since the gauge solution is not exact, the previous analysis is in fact overdone and is only meaningful to leading order in α' . If we solve the cubic equation (3.10) perturbatively, we find the turning point at

$$r_{\min} = \frac{L}{\sqrt{E^2 - 1}e^{\phi_0}} \left[1 - \frac{1}{2}Qe^{2\phi_0} \frac{\frac{L^2}{(E^2 - 1)e^{2\phi_0}} + 2\lambda^2}{\left\{ \frac{L^2}{(E^2 - 1)e^{2\phi_0}} + \lambda^2 \right\}^2} \right] + \mathcal{O}(Q^2). \quad (4.1)$$

Note that $r_{\min} > |ae^{-\phi_0}|$ for $\lambda \neq 0$. Thus there is always a turning point in this case.

Now to find out the structure of the orbits let us reexamine the effective potential eq.(3.9) only up to $\mathcal{O}(Q)$:

$$V^2 = e^{-2\phi_0} + \frac{L^2}{r^2}e^{-4\phi_0} + Qe^{-4\phi_0} \frac{r^2 + 2\lambda^2}{(r^2 + \lambda^2)^2} \left[(E^2 - 1) - \frac{2L^2}{r^2}e^{-2\phi_0} \right] + \mathcal{O}(Q^2). \quad (4.2)$$

⁴ Such an inverse cubic law appears in the electron-monopole case, but only in terms of the modified position vector. For more detail, see ref.[9].

Note that this potential does not have any critical points but is monotonically decreasing. As $r \rightarrow \infty$, $V^2 \rightarrow e^{-2\phi_0}$ regardlessly of the angular momentum. In the $r \rightarrow 0$ limit the situation depends on the angular momentum. If $L \neq 0$, $V^2 \rightarrow \infty$ as $r \rightarrow 0$, but if $L = 0$, $V^2 \rightarrow e^{-2\phi_0} + 2Qe^{-4\phi_0}(E^2 - 1)/\lambda^2$ as $r \rightarrow 0$. Since $\tilde{E}^2 \geq e^{-2\phi_0}$, all the orbits are open except for a trivial circular orbit.

Since the solution we have at hand is valid only for $\lambda \gg \sqrt{\alpha'}$, it is not surprising that in the limit $\lambda \rightarrow 0$ we do not recover the limit of the symmetric solution case, which will be studied in the next section. In other words, the double limit $r \rightarrow 0$ and $\lambda \rightarrow 0$ is not order independent.

We can attempt to solve the equation of motion eq.(3.12) perturbatively. To order Q we obtain

$$u = \frac{\sqrt{E^2 - 1}e^{\phi_0}}{L} \cos(\varphi - \varphi_0) + Qv, \quad (4.3)$$

where v satisfies the differential equation

$$\begin{aligned} \sin(\varphi - \varphi_0) \frac{dv}{d\varphi} &= v \cos(\varphi - \varphi_0) \\ &- \frac{\frac{1}{2} \sqrt{E^2 - 1}}{Le^{\phi_0}} \frac{\frac{E^2 - 1}{L^2} e^{2\phi_0} \cos^2(\varphi - \varphi_0) + 2\lambda^2 \left\{ \frac{E^2 - 1}{L^2} e^{2\phi_0} \cos^2(\varphi - \varphi_0) \right\}^2}{\left\{ 1 + \lambda^2 \frac{E^2 - 1}{L^2} e^{2\phi_0} \cos^2(\varphi - \varphi_0) \right\}^2}. \end{aligned} \quad (4.4)$$

Note that the leading order term in eq. (4.3) describes simply a straight line so that it is clear that with the correction the solution does not show conic motion.

From eqs.(3.13)(3.14) we can compute the components of the acceleration up to $\mathcal{O}(Q^2)$ as follows:

$$a_r = Q \frac{r(r^2 + 3\lambda^2)A_2}{e^{4\phi_0}(r^2 + \lambda^2)^3} - 2Q^2 \frac{r(r^2 + 2\lambda^2)(r^2 + 3\lambda^2)A_3}{e^{6\phi_0}(r^2 + \lambda^2)^5} + \mathcal{O}(Q^3), \quad (4.5)$$

and

$$a_\varphi = \pm Q \frac{2L(r^2 + 3\lambda^2)}{e^{5\phi_0}(r^2 + \lambda^2)^3} A_1^{1/2} \mp Q^2 \frac{L(r^2 + 2\lambda^2)(r^2 + 3\lambda^2)}{e^{6\phi_0}(r^2 + \lambda^2)^5} \left[4 + \frac{A_2}{e^{\phi_0} A_1^{1/2}} \right] + \mathcal{O}(Q^3), \quad (4.6)$$

where \pm is determined by the sign of \dot{r} and $A_n \equiv (E^2 - 1) - n \frac{L^2}{r^2 e^{2\phi_0}}$.

The radial position r_a at which $a_r = 0$ is given by

$$r_a = \frac{\sqrt{2}L}{\sqrt{E^2 - 1}e^{\phi_0}} \left[1 + \frac{1}{2}Q(E^2 - 1) \frac{\frac{2L^2}{(E^2 - 1)e^{2\phi_0}} + 2\lambda^2}{\left\{ \frac{2L^2}{(E^2 - 1)e^{2\phi_0}} + \lambda^2 \right\}^2} \right] + \mathcal{O}(Q^2). \quad (4.7)$$

Note that $r_a > r_{\min}$ and $a_r > 0$ ($a_r < 0$) for $r > r_a$ ($r < r_a$).

Using the ratio

$$\frac{a_\varphi}{a_r} = \pm \frac{2L}{re^{\phi_0}} \frac{A_1^{1/2}}{A_2} \left[1 + \tilde{Q} \frac{(r^2 + 2\lambda^2)}{(r^2 + \lambda^2)^2} \left(2 \frac{A_3}{A_2} - \frac{1}{2L} \frac{A_2}{A_1} \right) \right] + \mathcal{O}(Q^2), \quad (4.8)$$

we have

$$\left| \frac{a_\varphi}{a_r} \right| > \left| \frac{rd\varphi}{dr} \right|, \quad (4.9)$$

from which it follows that the trajectory always bends concave (i.e. inward) with respect to the origin.

There are no closed orbits except a trivial unstable circular orbit with radius $r = r_{\min}$. If the radial component of the velocity of the string is initially directed towards the fivebrane, it will spiral in to the turning point and then spiral away to infinity. If the string is directed away from the fivebrane, it will spiral away to infinity in a concave trajectory.

5. Symmetric Solution

For the symmetric solution we now keep the exact form of the generic case and set $\lambda = 0$. For $L = 0$ we have radial motion, and there is no turning point, as in the elementary fivebrane case[10]. The force is always repulsive and the attractive force region shrinks down to zero.

For $L \neq 0$ the cubic equation reduces to a linear equation leading to a turning point at

$$r_{\min} = \tilde{a}, \quad (5.1)$$

provided that $L^2 > (E^2 - 1)Q$. If $L^2 \leq (E^2 - 1)Q$, then $r_{\min} = 0$.

If initially the string is at r_{\min} with $\dot{r} = 0$, for constant $\dot{\varphi}$ there is an unstable circular orbit. For given E and L the size of this circular orbit depends only on the instanton charge Q of the fivebrane source. Since Q is quantized in terms of α' , r_{\min} is discretized accordingly. This is also analogous to the circular orbit of an electron around a monopole, in which the radius is discretized by the monopole charge.

In the symmetric solution case the effective potential can be simplified to

$$V^2 = E^2 e^{-2\phi_0} - \frac{E^2 - 1}{e^{2\phi_0} + \frac{Q}{r^2}} + \frac{L^2}{r^2 \left[e^{2\phi_0} + \frac{Q}{r^2} \right]^2}. \quad (5.2)$$

In the domain $r \geq r_{\min}$ the potential does not have any critical points but is monotonically decreasing. As $r \rightarrow \infty$, $V^2 \rightarrow e^{-2\phi_0}$ regardlessly of the angular momentum, as in the gauge solution. This implies that the long distance behavior is similar to the gauge solution case. In other words, we cannot distinguish the two cases by looking at the long-distance behaviour. The $r \rightarrow 0$ limit of V^2 , however, does not depend on the angular momentum in this case, so that $V^2 \rightarrow E^2 e^{-2\phi_0}$ as $r \rightarrow 0$.

We now rewrite the orbit equation as

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{E^2 - 1}{L^2} r^4 \left[e^{2\phi_0} + \frac{Q}{r^2} \right] - r^2, \quad (5.3)$$

which can be simplified further using a as

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{r^2}{a^2 + Q} (r^2 e^{2\phi_0} - a^2). \quad (5.4)$$

Again in analogy with the Keplerian analysis of Newtonian dynamics we reparametrize by $u \equiv 1/r$ and obtain

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{1}{a^2 + Q} (e^{2\phi_0} - a^2 u^2). \quad (5.5)$$

This equation of motion can be easily solved with the result

$$u = \frac{1}{r} = \frac{1}{\tilde{a}} \cos(\omega(\varphi - \varphi_0)) \quad \text{for } a^2 > 0, \quad (5.6a)$$

$$u = \frac{1}{r} = \frac{1}{i\tilde{a}} \cosh(\omega(\varphi - \varphi_0)) \quad \text{for } a^2 \leq 0, \quad (5.6b)$$

where $\omega^2 = |a^2|/(a^2 + Q)$ and $\tilde{a} = a e^{-\phi_0}$ as before. The solution for the $a^2 = 0$ case $u = \pm e^{\phi_0} Q^{-1/2} (\varphi - \varphi_\infty)$ arises as a limiting case of both (5.6a) and (5.6b). Note that in both cases the orbits are not conic! Thus we do not recover any of the orbits of Newtonian dynamics. Even if we compare the above orbits with those of Schwarzschild geometry, it is easy to see that there are no similarities.

The above solutions (5.6a) and (5.6b) correspond to two qualitatively different classes of orbits determined from the initial conditions. For both cases, it follows from the monotonicity of the effective potential in the range $r \geq r_{\min}$ that if the test string is directed initially away from the fivebrane (i.e. the radial component of the initial velocity is positive), then it will spiral away to infinity. If the string is directed towards the fivebrane, then in the two cases the motion is as follows: If $L^2 \leq (E^2 - 1)Q$ ($a^2 \leq 0$) as in eq. (5.6b), the string spirals into the fivebrane in an infinite amount of proper time, thus never

observing a singularity (the special case of $L = 0$ for the elementary fivebrane was discussed in [10]). If $L^2 > (E^2 - 1)Q$ ($a^2 > 0$) as in eq. (5.6a), there is a point of closest approach $r_{\min} = \tilde{a}$, after which the string swings back to infinity. A geometrical way to distinguish the two cases from the initial conditions is to draw a 3-dimensional cone in the transverse four space with vertex at the string, axis along the radial direction and half-angle $\text{Arctan}(\sqrt{Q}/r)$. If the velocity vector (either proper or coordinate) lies within the cone then $a^2 \leq 0$. Otherwise, $a^2 > 0$.

As in the previous two sections, we compute the components of the acceleration of the test string in order to study the competition between the attractive and repulsive forces. The radial component is given by

$$a_r = \frac{Q}{r^5 \left[e^{2\phi_0} + \frac{Q}{r^2} \right]^3} \left[(E^2 - 1)r^2 \left\{ e^{2\phi_0} + \frac{Q}{r^2} \right\} - 2L^2 \right] \quad (5.7)$$

and the angular component is

$$a_\varphi = \pm \frac{2QL}{r^5 \left[e^{2\phi_0} + \frac{Q}{r^2} \right]^3} \left[(E^2 - 1)r^2 \left(e^{2\phi_0} + \frac{Q}{r^2} \right) - L^2 \right]^{1/2}, \quad (5.8)$$

where the sign is chosen according to the sign of \dot{r} . Again dividing by E^2 , we can obtain the components of the acceleration measured by a distant observer.

Again we have $a_r > 0$ ($a_r < 0$) if $r > r_a$ ($r < r_a$), where here $r_a^2 = \left(\frac{L^2}{E^2 - 1} + a^2 \right) e^{-2\phi_0} > r_{\min}^2$ for $L > 0$. Recall that for radial motion $r_a = 0$ and $a_r > 0$ always and the “force” is always outward, i.e. repulsive. Usually the angular momentum signals the existence of an outward force region, but here we have an extra inward force region $r_a > r > r_{\min}$ because of the nonzero angular momentum, as discussed before. The source of this attractive force is the instanton at the center. This phenomenon is not completely unknown. For example, in the electron-monopole case the angular motion of the electron generates a current which interacts with the monopole charge to generate an extra force.

It is easy to see from the acceleration that for both types of orbits we have

$$\left| \frac{a_\varphi}{a_r} \right| > \left| \frac{r d\varphi}{dr} \right|, \quad (5.9)$$

from which it follows that the trajectory always bends concave (i.e. inward) with respect to the origin. Another way to see this is to note that for the $L^2 > (E^2 - 1)Q$ ($a^2 > 0$)

case, $\Delta\varphi = \pi(e^{2\phi_0} + Q/a^2)^{1/2} > \pi$, where $\Delta\varphi$ is the total angular deviation. In fact, one can determine from the initial conditions the number of loops the string makes around the fivebrane before heading off to infinity. n loops means $2\pi n < \Delta\varphi \leq 2\pi(n+1)$ which is equivalent to

$$\left(1 - \frac{1}{4n^2}\right) \left(\frac{L^2}{E^2 - 1}\right) < Q \leq \left(1 - \frac{1}{4(n+1)^2}\right) \left(\frac{L^2}{E^2 - 1}\right). \quad (5.10)$$

Note that as $n \rightarrow \infty$, $a \rightarrow 0$ and the string spirals towards the fivebrane. The “swingshot” orbits for $a^2 > 0$ are analogous to electron-monopole orbits in which the electron swings around the monopole before heading off to infinity.

6. Discussion

For nonzero angular momentum L the motion of the test string around a generic heterotic fivebrane source in the transverse space is either circular or open. The open orbits spiral in to the turning point, then spiral away to infinity. The radius of the circular orbit is governed by the quantized instanton charge Q and is discretized accordingly, in analogy with the radius of the circular orbit of an electron around a monopole, which is governed by the quantized monopole charge. Other analogies with the electron-monopole system include the shifting of the angular momentum of the string due to the fivebrane, and the existence of open swingshot orbits.

Furthermore, the open orbits are not conic. This signifies that the string motion around the fivebrane differs fundamentally from previously studied orbits in gravitational theory. In fact, the radial component of the acceleration goes as Qr^{-3} asymptotically as $r \rightarrow \infty$. This implies that the force does not satisfy the inverse square law, in direct contrast with General Relativity. The source of this departure lies partially in the presence of the antisymmetric tensor in the string-fivebrane system, which generates an additional repulsive force which dominates the dynamics at long distances and precludes the existence of stable bounded orbits.

Although our analysis has been done in the transverse space of the fivebrane only, the implications of our findings could be significant. If these structures survive compactification, one might be able to formulate an interesting test for string theory. Either there is no remnant of the fivebrane after compactification if we fail to observe these new structures, or string theory should lead to structures different from those of Newtonian dynamics at distances longer than the Planck scale, yet at a still sufficiently short scale (otherwise, this structure may not be observable anyhow). It is therefore important to seriously investigate the compactifications of the fivebrane solutions of ten-dimensional string theory.

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References

- [1] For recent reviews, see M.J. Duff and J.X. Lu, “A Duality between Strings and Five-branes,” Texas A&M preprint, CTP-TAMU-28/91 (1991); C.G. Callan, “Instantons and Solitons in Heterotic String Theory,” Princeton preprint, PUPT-1278 (1991); C.G. Callan, J.A. Harvey and A. Strominger, “Supersymmetric String Solitons,” Chicago preprint, EFI-91-66 (1991); and references therein.
- [2] M.J. Duff, *Class. Quan. Grav.* **5** (1988) 189;
M.J. Duff, in *Superworld II*, ed. by A. Zichichi (Plenum, New York, 1990).
- [3] A. Strominger, *Nucl. Phys.* **B343** (1990) 167.
- [4] C. Montonen and D. Olive, *Phys. Lett.* **72B** (1977) 117.
- [5] C.G. Callan, J.A. Harvey and A. Strominger, *Nucl. Phys.* **B359** (1991) 611.
- [6] M.J. Duff and J. X. Lu, *Nucl. Phys.* **B354** (1991) 141.
- [7] M. Green and J.H. Schwarz, *Phys. Lett.* **151B** (1985) 21;
R.R. Khuri, *Phys. Lett.* **259B** (1991) 261;
C. Callan, J. Harvey and A. Strominger, *Nucl. Phys.* **B367** (1991) 60.
- [8] E.A. Bergshoeff and M. de Roo, *Nucl. Phys.* **B328** (1989) 439.
- [9] D.G. Boulware, L.S. Brown, R.N. Cahn, S.D. Ellis and C. Lee, *Phys. Rev. D* **14** (1976) 2708.
- [10] M.J. Duff, R.R. Khuri and J.X. Lu, Texas A&M preprint, CTP-TAMU 89/91 (1991).